Quasi-arithmetic Gauss-type iteration

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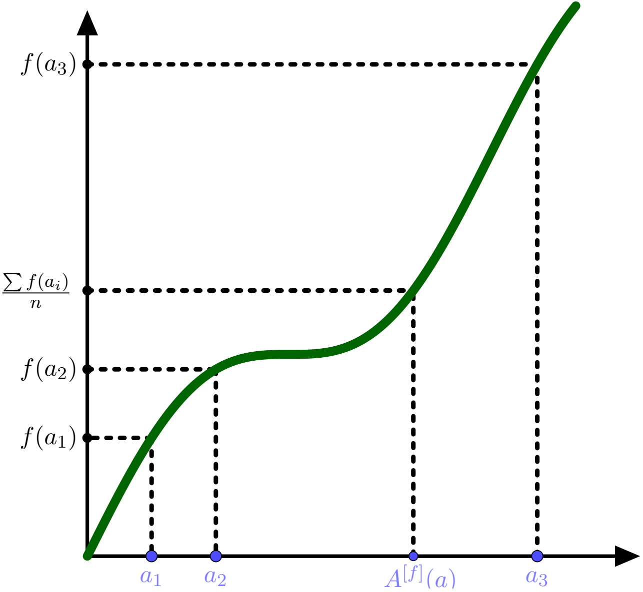
Quasi-arithmetic mean

Quasi-arithmetic mean is defined For any continuous strictly monotone function $f\colon I \to \mathbb{R}$ (I – an open interval). Define a *quasi-arithmetic mean* $A^{[f]}\colon \bigcup_{n=1}^{\infty} I^n \to I$ by

$$A^{[f]}(a) := f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(a_i)\right),$$

where $n \in \mathbb{N}$ and $a \in I^n$.



$f(a_3)$
$\frac{\sum f(a_i)}{n}$
$f(a_2)$
$f(a_1)$
a_1
a_2
$A^{[f]}(a)$
a_3

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Let $\mathcal{S}(I)$ be a family of $\mathcal{C}^2$ functions defined on an interval I, with nowhere vanishing first derivative and the second derivative with [locally] bounded variation.
Roughly: smooth enought.

Theorem 1 ([Pasteczka, 2016], Lemma 4.1)

Let I be an interval, $f \in \mathcal{S}(I)$. For a given vector $a \in I^n$ for some $n \in \mathbb{N}$ one has

$$A^{[f]}(a) = \overline{a} + \tfrac{1}{2}\operatorname{Var}(a) \cdot \frac{f''(\overline{a})}{f'(\overline{a})} + R_f(a) + R_f^*(a),$$

where

$$R_f(a) := \frac{1}{2n \cdot f'(\overline{a})} \sum_{i=1}^{n} \int_{\overline{a}}^{a_i} (a_i - t)^2 df''(t),$$

$$R_f^*(a) := \int_{\overline{a}}^{A^{[f]}(a)} \frac{\big(f(t) - f(A^{[f]}(a))\big) f''(t)}{f'(t)^2}\, dt.$$

Let

$$\mathcal{S}_K(I) := \Big\{ f \in \mathcal{S}(I) \colon \; \big\| f''/f' \big\|_\infty \le K \Big\};$$
$$\mathcal{S}^{Lip}(I) := \Big\{ f \in \mathcal{S}(I) \colon \; f'' \text{ is Lipschitz} \Big\};$$
$$\mathcal{S}^{Lip}_K(I) := \mathcal{S}^{Lip}(I) \cap \mathcal{S}_K(I).$$

Lemma 1 ([Pasteczka, 2016], Lemma 4.2)

For every $f \in \mathcal{S}_1(I)$,

$$|R_f(a)| \le \frac{1}{6n} \cdot \exp(\|f''/f'\|_1) \cdot \sum_{i=1}^{n} |a_i - \overline{a}|^3 \,,$$

$$\left|R_f^*(a)\right| \le (A^{[f]}(a) - \overline{a})^2 \cdot \exp(\left\|f''/f'\right\|_1).$$

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Lemma 2 ([Pasteczka, 2016], Lemma 4.3)

For every $f \in \mathcal{S}_1(I)$,

$$\left|A^{[f]}(a) - \overline{a}\right| \le \frac{3+7e}{6}(\max a - \min a)^2.$$

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Let I be an interval, $f, g \in \mathcal{S}_1(I)$ and $a \in I^k$ for some $k \in \mathbb{N}$.

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2. $\Big|A^{[f]}(a) - A^{[g]}(a)\Big| \le \frac{3+7e}{3} \cdot (\max a - \min a)^2$.

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Consequently, for $f, g \in \mathcal{S}_1(I)$ we have

$$\Big|A^{[f]}(a) - A^{[g]}(a)\Big| \le \Theta(\max a - \min a),$$

where $\Theta \colon \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$\Theta(x) := \min\Big(\ln(\tfrac{e^x+1}{2}), \tfrac{3+7e}{3} \cdot x^2\Big).$$

Theorem 2

Let $K > 0$, $\mathbf{f} = (f_1, \dots, f_n) \in \mathcal{S}_K^{Lip}(I)^n$ *and* $a \in I^n$.
Consider the mapping $\mathbf{A}_{[\mathbf{f}]} := (A^{[f_1]}, \dots, A^{[f_n]})\colon I^n \to I^n$. *Then, for all* $k \in \mathbb{N}$,

$$\max \mathbf{A}_{[\mathbf{f}]}^k(a) - \min \mathbf{A}_{[\mathbf{f}]}^k(a) \le \tfrac{1}{K} \cdot \Theta^k(K \cdot (\max a - \min a)),$$

where $\Theta\colon \mathbb{R}_+ \to \mathbb{R}_+$ *is defined by*

$$\Theta(x) := \min\left(\ln(\tfrac{e^x+1}{2}), \tfrac{3+7e}{3} \cdot x^2 \right).$$

Estimation of Θ^k $\quad \Theta(x) = \min\left(\ln(\frac{e^x+1}{2}), \frac{3+7e}{3} \cdot x^2\right)$

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4. $\Theta^k(x) \le \ln\frac{17}{16} < \frac{1}{10}$ for all $x\in[0,\infty)$ and $k \ge \lceil 4+\frac{x}{\ln 2}\rceil$;

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6. $\Theta^k(x) \le \frac{3}{3+7e}\cdot\left(\frac{3+7e}{30}\right)^{2^{k-\lceil 4+x/\ln 2\rceil}}$ for $x\ge 0$; $k\ge\lceil 4+\frac{x}{\ln 2}\rceil$.

Corollary 1

Let $K > 0$, $\mathbf{f} = (f_1, \dots, f_n) \in \mathcal{S}_K^{Lip}(I)^n$ *and* $a \in I^n$.
Consider the mapping $\mathbf{A}_{[\mathbf{f}]} := (A^{[f_1]}, \dots, A^{[f_n]}) \colon I^n \to I^n$. *Then, for all* $k \ge k_0 := \lceil 4 + \frac{K}{\ln 2} \cdot (\max a - \min a) \rceil$,

$$\max \mathbf{A}_{[\mathbf{f}]}^k(a) - \min \mathbf{A}_{[\mathbf{f}]}^k(a) \le \frac{3}{(3+7e)K} \cdot \left(\frac{3+7e}{30}\right)^{2^{k-k_0}}.$$

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Remark 1 of 3

This result provides an effective double-exponential estimation of the difference.
Right-hand side of an inequality **does not** depend on $\mathbf{f}$ (only implicitly, by K).

Corollary 1

Let $K>0$, $\mathbf{f}=(f_1,\dots,f_n)\in\mathcal{S}_K^{Lip}(I)^n$ and $a\in I^n$. Consider the mapping $\mathbf{A}_{[\mathbf{f}]}:=(A^{[f_1]},\dots,A^{[f_n]})\colon I^n\to I^n$. Then, for all $k\ge k_0:=\lceil 4+\frac{K}{\ln 2}\cdot(\max a-\min a)\rceil$,

$$\max \mathbf{A}_{[\mathbf{f}]}^k(a)-\min \mathbf{A}_{[\mathbf{f}]}^k(a)\le \frac{3}{(3+7e)K}\cdot\Big(\frac{3+7e}{30}\Big)^{2^{k-k_0}}.$$

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By this corollary we can deliver an upper bound of iterates which are sufficient to get appropriate precision of invariant mean, which is very useful for numerical calculations.

Corollary 1

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Remark 3 of 3

In the original paper [Pasteczka, 2016] this result appears with a different constants and with more complicated proof.

Lemma 4

For every $f \in \mathcal{S}_1^{Lip}(I)$,

$$|R_f(a)| \le \frac{\mathrm{Lip}(f'')}{2\,|f'(\overline{a})|} \cdot \delta(a)\,\mathrm{Var}(a)$$

$$|S_f(a)| \le \frac{\alpha^2}{4} \exp(\|f''/f'\|_1)\delta(a)^4.$$

Notice that these terms are $\mathcal{O}(\delta(a)^3)$. Thus if I is bounded then there exists a constant $E_f \in \mathbb{R}_+$ such that

$$A^{[f]}(a) = \overline{a} + \tfrac{1}{2}\,\mathrm{Var}(a) \cdot \frac{f''(\overline{a})}{f'(\overline{a})} \pm E_f \cdot \delta(a)^3 \text{ for all } a \in \bigcup_{n=1}^{\infty} I^n\,.$$

In the paper [Pasteczka, 2016] the error was $\mathcal{O}(\delta(a)^2)$ and situation was completely different.

Important technical lemma

Lemma 5

Let I be a bounded interval, $f_1, \dots, f_n \in \mathcal{S}_1^{Lip}(I)$. Then

$$\frac{\operatorname{Var}_j\left(A^{[f_j]}(a)\right)}{\operatorname{Var}(a)^2} = \tfrac{1}{4}\operatorname{Var}_j\left(\frac{f_j''(\overline{a})}{f_j'(\overline{a})}\right) + \mathcal{O}(\delta(a))$$

for all $a \in \bigcup_{k=1}^{\infty} I^k$.

Main result

Now we get the following

Theorem 3

Let $\mathbf{f} = (f_1, \dots, f_n) \in \mathcal{S}^{Lip}(I)^n$ *and* $a \in I^n$. *Consider the mapping* $\mathbf{A}_{[\mathbf{f}]} := (A^{[f_1]}, \dots, A^{[f_n]})\colon I^n \to I^n$. *Then either* $\mathbf{A}_{[\mathbf{f}]}^s(a)$ *is a constant vector for some* $s \in \mathbb{N}$ *or*

$$\lim_{k\to\infty} \frac{\operatorname{Var} \mathbf{A}_{[\mathbf{f}]}^{k+1}(a)}{\big(\operatorname{Var} \mathbf{A}_{[\mathbf{f}]}^{k}(a)\big)^2} = \frac{1}{4} \operatorname{Var}_j \Big(\frac{f_j''(\mathscr{M}(a))}{f_j'(\mathscr{M}(a))}\Big),$$

where $\mathscr{M}$ *is a unique* $\mathbf{A}_{[\mathbf{f}]}$*-invariant mean.*

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where $\mathscr{M}$ *is a unique* $\mathbf{A}_{[\mathbf{f}]}$*-invariant mean.*

To prove this theorem it is sufficient to: • scale the interval I to obtain all functions in $\mathcal{S}^{Lip}_1(I_0)$ [technical but straightforward],

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Let $\mathbf{f} = (f_1, \dots, f_n) \in \mathcal{S}^{Lip}(I)^n$ and $a \in I^n$. Consider the mapping $\mathbf{A}_{[\mathbf{f}]} := (A^{[f_1]}, \dots, A^{[f_n]}) \colon I^n \to I^n$. Then either $\mathbf{A}_{[\mathbf{f}]}^s(a)$ is a constant vector for some $s \in \mathbb{N}$ or

$$\lim_{k\to\infty} \frac{\operatorname{Var} \mathbf{A}_{[\mathbf{f}]}^{k+1}(a)}{\left(\operatorname{Var} \mathbf{A}_{[\mathbf{f}]}^{k}(a)\right)^2} = \frac{1}{4} \operatorname{Var}_j \left(\frac{f_j''(\mathscr{M}(a))}{f_j'(\mathscr{M}(a))}\right),$$

where $\mathscr{M}$ is a unique $\mathbf{A}_{[\mathbf{f}]}$-invariant mean.

To prove this theorem it is sufficient to: • scale the interval I to obtain all functions in $\mathcal{S}_1^{Lip}(I_0)$ [technical but straightforward], • put $a \leftarrow \mathbf{A}_{[\mathbf{f}]}^k(a)$ in the previous lemma, • take the limit.

Example 1 [log – exp means]

Let $I = \mathbb{R}$, $n \in \mathbb{N}$, $s \in \mathbb{R}^n$, and $\mathbf{f} = (f_1, \dots, f_n)$ be given by

$$f_k(x) = \begin{cases} \exp(s_k \cdot x) & \text{for } s_k \neq 0, \\ x & \text{for } s_k = 0. \end{cases} \qquad (k = 1, \dots, n)$$

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For all $a\in\mathbb{R}^n$ and $k\ge k_0(a):=\left\lceil 4+\frac{1}{\ln 2}\left\|s\right\|_\infty\cdot(\max a-\min a)\right\rceil$ we have

$$\max \mathbf{A}^k_{[\mathbf{f}]}(a)-\min \mathbf{A}^k_{[\mathbf{f}]}(a)\le \frac{3}{(3+7e)\|s\|_\infty}\cdot\left(\frac{3+7e}{30}\right)^{2^{k-k_0(a)}}.$$

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For all $a \in \mathbb{R}^n$ and $k \geq k_0(a) := \lceil 4 + \frac{1}{\ln 2} \|s\|_\infty \cdot (\max a - \min a) \rceil$ we have

$$\max \mathbf{A}^k_{[\mathbf{f}]}(a) - \min \mathbf{A}^k_{[\mathbf{f}]}(a) \leq \tfrac{3}{(3+7e)\|s\|_\infty} \cdot \left(\tfrac{3+7e}{30}\right)^{2^{k-k_0(a)}}.$$

Moreover, if s and $a \in \mathbb{R}^n$ are both nonconstant vectors then

$$\lim_{k\to\infty} \frac{\operatorname{Var} \mathbf{A}^{k+1}_{[\mathbf{f}]}(a)}{\left(\operatorname{Var} \mathbf{A}^k_{[\mathbf{f}]}(a)\right)^2} = \frac{\operatorname{Var}(s)}{4}.$$

Example 2 [Power means / Hölder means]

Let $I = \mathbb{R}_+$, $n \in \mathbb{N}$, $s \in \mathbb{R}^n$ be a nonconstant vector, and $\mathbf{p} = (p_1, \dots, p_n)$ be given by

$$p_k(x) = \begin{cases} x^{s_k} & \text{for } s_k \neq 0, \\ \ln(x) & \text{for } s_k = 0. \end{cases} \qquad (k = 1, \dots, n)$$

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Then

$$\lim_{k \to \infty} \frac{\operatorname{Var} \mathbf{A}_{[\mathbf{p}]}^{k+1}(a)}{\left(\operatorname{Var} \mathbf{A}_{[\mathbf{p}]}^{k}(a)\right)^2} = \frac{\operatorname{Var}(s)}{4 \cdot K(a)^2},$$

where K is a unique $\mathbf{A}_{[\mathbf{p}]}$-invariant mean.

Further developments

Conjecture

Theorems above remain valid without requirement of Lipschitz property of second derivatives.

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Conjecture

The property

$$\mathbf{A}^s_{[\mathbf{f}]}(a) \text{ is a nonconstant vector for all } s \in \mathbb{N}$$

is [in some sense] stable with respect to a.

For example: set of all such a-s is open (has other regularity properties).

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